Regularity of solutions of abstract linear evolution equations

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Abstract: In this paper, we study regularity of solutions to linear evolution equations of the form $dX + AX dt = F(t)dt$ in a Banach space H, where A is a sectorial operator in H and $A^{-\alpha}F\,(\alpha > 0)$ belongs to a weighted Hölder continuous function space. Similar results are obtained for linear evolution equations with additive noise of the form $dX + AX dt = F(t)dt + G(t)dW(t)$ in a separable Hilbert space H, where $W(t)$ is a cylindrical Wiener process. Our results are applied to a model arising in neurophysiology, which has been proposed by Walsh [18].

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Contents

1. Introduction and main results

In recent years, existence, uniqueness and regularity of solutions to stochastic partial differential equations have been extensively studied by many authors. These topics have been developed mainly by using three approaches, that is, the semigroup methods (see Da Prato-Zabczyk [5], Tạ-Yagi [15], Tạ [16], and references therein), the variational methods (see Rozovskii [11], Prévôt-Röckner [10], and references therein), and the martingale measure methods (see Walsh [18]). Among others, stochastic linear evolution equations have been studied by many authors in Hilbert or L_2 spaces (see Rozovskii [11], Da Prato et al. [4, 5], Tạ [14]), in weighted Sobolev L_p spaces (see Krylov-Lototsky [7]), in weighted Hölder spaces (see Mikulevicius [8]), and in M-type 2 Banach space (Brzeźniak [2], Tạ-Yagi [13], Tạ-Yamamoto-Yagi [17]).

In this paper, we shall study regularity of solutions to both deterministic and stochastic linear evolution equations whose coefficients belong to weighted Hölder continuous function spaces by using the semigroup methods. Our results can be applied to a class of stochastic partial differential equations such as the Zakai equation in the nonlinear filtering problem (see Rozovskii [11]), the stochastic heat equation (see Hairer [6]), Nagumo's equation, Hodgkin-Huxley equations (see [18, 5]), which can be approximated by the equation:

$$\frac{\partial X}{\partial t} = \Delta X - aX + F(t) + G(t)\dot{W}.$$

As a consequence, some results obtained by Walsh [18] for a model arising in neurophysiology are improved.

Let us introduce the framework in the deterministic case. We consider the Cauchy problem for a linear evolution equation

$$\begin{cases} dX + AX\,dt = F(t)dt, & 0 < t \leq T, \\ X(0) = \xi \end{cases} \tag{1.1}$$

in a Banach space H with norm $\|\cdot\|$, where $T > 0$, $A\colon \mathcal{D}(A) \subset H \to H$ is a densely defined, closed linear operator in H, and F is an H-valued addition external force.

Definition 1.1. Let F satisfy the condition

$$\int_0^t S(t-s)F(s)ds < \infty, \qquad\qquad t \in [0, T],$$

where $S(t)$ is the C_0-semigroup in H generated by the operator $(-A)$. Then the process X defined by

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s)ds, \qquad\qquad t \in [0, T]$$

is called a mild solution of (1.1).

(In Remark 2 below, we will give a discussion about a relation between weak, mild and strong solutions.)

When the initial value $\xi \in H$ is arbitrary and $F \in L^1([0, T]; H)$, the Cauchy problem (1.1) possesses a unique continuous mild solution. Meanwhile, when $\xi \in \mathcal{D}(A)$ and $F \in W^{1,p}([0, T]; H)$ with $p \geq 1$, the mild solution possesses the regularity

$$X \in \mathcal{C}^1([0, T]; H) \cap \mathcal{C}([0, T]; \mathcal{D}(A)) \cap W^{1,p}([0, T]; H),$$

where $W^{1,p}([0, T]; H)$ denotes the Sobolev space of all H-valued functions u on $[0, T]$ whose weak derivative u' belongs to $L^p([0, T]; H)$ (see e.g., [1, 5]).

When $(-A)$ is a sectorial operator (i.e. it satisfies conditions (H1) and (H2) below), it generates an analytical semigroup $S(t)$. In [9], the author considered

the case $\xi = 0$ and $F \in \mathcal{C}^\alpha([0,T]; H)$ for some $\alpha \in (0,1)$, where $\mathcal{C}^\alpha([0,T]; H)$ denotes the space of α-Hölder continuous functions on $[0,T]$. Then the mild solution of (1.1) satisfies

$$X \in \mathcal{C}^1([0,T]; H) \cap \mathcal{C}([0,T]; \mathcal{D}(A)).$$

In [12], under the assumption $F(0) \in \mathcal{D}_A(\alpha, \infty)$, it has been verified that

$$X \in \mathcal{C}^{1,\alpha}([0,T]; H) \cap \mathcal{C}^\alpha([0,T]; \mathcal{D}(A)),$$

where $\mathcal{C}^{1,\alpha}$ is the space of $\mathcal{C}^1$ functions with derivatives in $\mathcal{C}^\alpha$. If F is taken from $\mathcal{C}([0,T]; \mathcal{D}_A(\alpha, \infty))$ then (see [3])

$$X \in \mathcal{C}^1([0,T]; \mathcal{D}_A(\alpha, \infty)) \cap \mathcal{C}([0,T]; \mathcal{D}_A(\alpha + 1, \infty)),$$

where $\mathcal{D}_A(\alpha, \infty)$ is the space of all $x \in H$ such that $\sup_{t>0} \frac{\|S(t)x - x\|}{t^\alpha} < \infty$, and $\mathcal{D}_A(\alpha + 1, \infty) = \{x \in \mathcal{D}(A) \colon Ax \in \mathcal{D}_A(\alpha, \infty)\}$. When $\xi \in H$ is arbitrary and F belongs to a weighted Hölder continuous function space $\mathcal{F}^{\beta,\sigma}((0,T]; H)$ (see the definition below), Yagi [19] showed that

$$X \in \mathcal{C}^1((0,T]; H) \cap \mathcal{C}([0,T]; H) \cap \mathcal{C}((0,T]; \mathcal{D}(A)).$$

He also obtained the maximal regularity for both initial value $\xi \in \mathcal{D}(A^\beta)$ and external force function $F \in \mathcal{F}^{\beta,\sigma}((0,T]; H)$ (see [20]-[22]):

$$A^\beta X \in \mathcal{C}([0,T]; H),$$

$$\frac{dX}{dt}, AX \in \mathcal{F}^{\beta,\sigma}((0,T]; H).$$

In the present paper, we assume that $A^{-\alpha} F$ belongs to the weighted Hölder continuous function space $\mathcal{F}^{\beta,\sigma}((0,T]; H)$ with some positive α, i.e. F satisfies both temporal and spatial regularity. We will show both temporal and spatial regularity of solutions to (1.1). Similar results are obtained for linear evolution equations in Hilbert spaces with additive noise. Our result will be applied to a model arising in neurophysiology.

Let us review the function space $\mathcal{F}^{\beta,\sigma}((0,T]; H)$ for two exponents $0 < \sigma < \beta \leq 1$ (see [22]). The space $\mathcal{F}^{\beta,\sigma}((0,T]; H)$ consists of all H-valued continuous functions $f(t)$ on $(0,T]$ (resp. $[0,T]$) when $0 < \beta < 1$ (resp. $\beta = 1$) with the following three properties:

(i) When $\beta < 1$,

$$t^{1-\beta} f(t) \text{ has a limit as } t \to 0. \tag{1.2}$$

(ii) The function f is Hölder continuous with the exponent σ and with the weight $s^{1-\beta+\sigma}$, i.e.

$$\sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma} \|f(t) - f(s)\|}{(t-s)^\sigma}$$
$$= \sup_{0 \leq t \leq T} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma} \|f(t) - f(s)\|}{(t-s)^\sigma} < \infty. \tag{1.3}$$

(iii)
$$\lim_{t \to 0} w_f(t) = 0,$$

$$\text{where } w_f(t) = \sup_{0 \le s < t} \frac{s^{1-\beta+\sigma} \|f(t) - f(s)\|}{(t-s)^\sigma}. \tag{1.4}$$

Then $\mathcal{F}^{\beta,\sigma}((0,T];H)$ becomes a Banach space with norm

$$\|f\|_{\mathcal{F}^{\beta,\sigma}((0,T];H)} = \sup_{0 \le t \le T} t^{1-\beta} \|f(t)\| + \sup_{0 \le s < t \le T} \frac{s^{1-\beta+\sigma} \|f(t) - f(s)\|}{(t-s)^\sigma}.$$

For simplicity, if not specified the norm of f in $\mathcal{F}^{\beta,\sigma}((0,T];H)$ will be denoted by $\|f\|_{\mathcal{F}^{\beta,\sigma}}$. The following useful inequality follows the definition directly. For every $f \in \mathcal{F}^{\beta,\sigma}((0,T];H), 0 < s < t \le T$ we have

$$\begin{cases} \|f(t)\| \le \|f\|_{\mathcal{F}^{\beta,\sigma}} t^{\beta-1}, \\ \|f(t) - f(s)\| \le w_f(t)(t-s)^\sigma s^{\beta-\sigma-1} \le \|f\|_{\mathcal{F}^{\beta,\sigma}} (t-s)^\sigma s^{\beta-\sigma-1}. \end{cases} \tag{1.5}$$

In addition, it is not hard to show that

$$\mathcal{F}^{\gamma,\sigma}((0,T];H) \subset \mathcal{F}^{\beta,\sigma}((0,T];H), \qquad 0 < \sigma < \beta < \gamma \le 1. \tag{1.6}$$

Remark 1. The space $\mathcal{F}^{\beta,\sigma}((0,T];H)$ is not a trivial space. When $0 < \sigma < \beta < 1$, $f(t) = t^{\beta-1}g(t) \in \mathcal{F}^{\beta,\sigma}((0,T];H)$, where $g(t)$ is any H-valued function on $[0,T]$ such that $g \in \mathcal{C}^\sigma([0,T];H)$ and $g(0) = 0$. When $0 < \sigma < \beta = 1$, the space $\mathcal{F}^{1,\sigma}((0,T];H)$ includes the space of all Hölder continuous functions with the exponent σ.

Let us now formulate the precise conditions on the coefficients in (1.1).

(H1) The spectrum $\sigma(A)$ of A is contained in an open sectorial domain Σ_ϖ:

$$\sigma(A) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \varpi\}, \qquad 0 < \varpi < \frac{\pi}{2}.$$

(H2) The resolvent satisfies the estimate

$$\|(\lambda - A)^{-1}\| \le \frac{M_\varpi}{|\lambda|}, \qquad \lambda \notin \Sigma_\varpi$$

with some constant $M_\varpi > 0$ depending only on the angle ϖ.

(H3)
$$A^{-\alpha}F \in \mathcal{F}^{\beta,\sigma}((0,T];H) \qquad \text{with } 0 < \sigma < \beta \le 1 \text{ and}$$
$$\frac{1+\sigma}{4} < \alpha \le \frac{\beta}{2}.$$

Under the assumptions (H1) and (H2), it is well-known that

Proposition 1 (see e.g., [22]). *Let* (H1) *and* (H2) *be satisfied. Then*

(i) $(-A)$ *generates a semigroup* $S(t) = e^{-tA}$, *i.e.* $S(t-s)$ *enjoys the property*

$$\begin{cases} S(t+s) = S(t)S(s), & 0 \le t, s < \infty. \\ S(0) = I. \end{cases}$$

(ii) *For every* $t > 0$ *and* $\theta \ge 0$

$$\begin{aligned} &\|A^\theta S(t)\| \le \iota_\theta t^{-\theta}, \\ &\text{where } \iota_\theta := \sup_{0 \le t < \infty} t^\theta \|A^\theta S(t)\| < \infty. \end{aligned} \tag{1.7}$$

In particular,
$$\|S(t)\| \le \iota_0, \qquad\qquad 0 \le t < \infty. \tag{1.8}$$

(iii) *For every* $\theta > 0$ *there exists a constant* υ_θ *such that*

$$\|A^{-\theta}\| \le \upsilon_\theta. \tag{1.9}$$

(iv) *For every* $0 < \theta \le 1$

$$t^\theta A^\theta S(t) \text{ converges to } 0 \text{ strongly on } H \text{ as } t \to 0. \tag{1.10}$$

From now when we use notations ι_θ or υ_θ, it refers to constants in (1.7) or (1.9), respectively. Now we can state the main result for (1.1).

Theorem 1.2. *Let* (H1), (H2) *and* (H3) *be satisfied. Let* ξ *take values in* $\mathcal{D}(A^\beta)$. *Then there exists a unique mild solution of* (1.1) *possessing the regularity:*

$$X \in \mathcal{C}((0,T]; \mathcal{D}(A^{1-\alpha})),$$

$$A^\alpha X \in \mathcal{C}([0,T]; H) \cap \mathcal{F}^{\beta-\sigma+\gamma, \gamma}((0,T]; H) \qquad \text{for every } \gamma \in [0, \sigma].$$

Furthermore, X *satisfies the estimate*

$$\|X(t)\| \le \iota_\alpha B(\beta, 1-\alpha) \|A^{-\alpha} F\|_{\mathcal{F}^{\beta,\sigma}} t^{\beta-\alpha} + \iota_0 \|\xi\|, \qquad t \in [0,T], \tag{1.11}$$

where $B(\cdot, \cdot)$ *is the beta function.*

Let us next formulate the analogous result for linear evolution equations with additive noise. We proceed to study the Cauchy problem for an abstract stochastic evolution equation

$$\begin{cases} dX + AXdt = F(t)dt + G(t)dW(t), & 0 < t \le T, \\ X(0) = \xi \end{cases} \tag{1.12}$$

in a separable Hilbert space H. Here, $W(t)$ is a cylindrical Wiener process on a separable Hilbert space U, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The coefficient G is a measurable process from $([0,T], \mathcal{B}([0,T]))$ to $(L_2(U;H), \mathcal{B}(L_2(U;H)))$, where $L_2(U;H)$ denotes the space of all Hilbert-Schmidt operators from U to H. The linear operator A and the function F are the same to ones in (1.1). And the initial value ξ is an $\mathcal{F}_0$-measurable random variable.

Definition 1.3. Let F and G satisfies the conditions

$$\int_0^T \|S(t-s)F(s)\|ds < \infty$$

and

$$\int_0^T \|S(t-s)G(s)\|_{L_2(U;H)}^2 ds < \infty.$$

Then an H-valued predictable process $X(t), t \in [0, T]$ defined by

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)G(s)dW(s)$$

is called a mild solution of (1.12).

Remark 2. In the semigroup approach to parabolic evolution equations, mild (and *weak*) solutions are mainly studied, because the existence of *strong* solutions (or *real* solutions) is very rare [5]. Let us review the notions of strong and weak solutions. Suppose that F and G satisfy the conditions

$$\int_0^T \|F(s)\|ds < \infty$$

and

$$\int_0^T \|G(s)\|_{L_2(U;H)}^2 ds < \infty.$$

Then an H-valued predictable process X on $[0, T]$ is called a strong solution to (1.12) if X takes values in $\mathcal{D}(A)$ a.s. and satisfies

$$\int_0^T \|AX(s)\|ds < \infty, \qquad\qquad \text{a.s.}$$

and almost surely

$$X(t) = \xi + \int_0^t [F(s) - AX(s)]ds + \int_0^t G(s)dW(s), \qquad\qquad t \in [0, T].$$

The process X is called a weak solution if for every $h \in \mathcal{D}(A^*)$ and $t \in [0, T]$ we have

$$\langle X(t), h \rangle = \langle \xi, h \rangle + \int_0^t [\langle F(s), h \rangle - \langle X(s), A^*h \rangle]ds$$

$$+ \int_0^t \langle G(s), h \rangle dW(s).$$

It is clear that a strong solution is also a weak solution. In addition, a weak solution is a mild solution [5]. The inverses are not true in general. They can take place, however, under some special conditions. In [5], the authors showed

that a mild solution becomes a weak solution under some conditions on G. In [**?**], under the assumption

$$\|AS(t)\| \leq Ct^{-\delta}, \qquad\qquad t > 0$$

with some constants $C > 0$ and $\delta \in (0, \frac{1}{2})$, we proved that a mild solution is also a strong solution.

We suppose that the process G in the equation (1.12) belongs to a weighted Hölder continuous function space:

(H4) $\qquad\qquad \mathrm{G} \in \mathcal{F}^{\beta,\sigma}((0,\mathrm{T}]; \mathrm{L}_2(\mathrm{U}; \mathrm{H})) \qquad$ with $0 < \sigma < \beta - \dfrac{1}{2}$.

Our results for the stochastic case are as follows. In the case there is no external force, we have

Theorem 1.4. *Assume that $F \equiv 0$ on $[0, T]$. Let* (H1)*,* (H2) *and* (H4) *be satisfied. Let ξ take values in $\mathcal{D}(A^\beta)$ a.s. such that $\mathbb{E}\|A^\beta \xi\| < \infty$. Then there exists a unique mild solution of* (1.12) *possessing the regularity:*

$$X \in \mathcal{C}((0,T]; \mathcal{D}(A^\nu)), \quad A^{\alpha_1} X \in \mathcal{C}^\gamma([\epsilon, T]; H) \qquad a.s.,$$

and

$$\mathbb{E}\|A^{\alpha_1} X\| \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R})$$

for every $0 < \nu < \frac{1}{2}, 0 < \alpha_1 \leq \frac{1}{2} - \sigma, 0 < \gamma < \sigma$ and $\epsilon \in (0, T]$. Furthermore, X satisfies the estimate

$$\mathbb{E}\|X(t)\| \leq C[\mathbb{E}\|A^\beta \xi\| + \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))} t^{\beta - \frac{1}{2}}], \qquad t \in [0, T], \quad (1.13)$$

where C is some constant depending only on the exponents and constants ι_θ, υ_θ ($\theta \geq 0$).

In general, we have

Theorem 1.5. *Let* (H1)*,* (H2)*,* (H3) *and* (H4) *be satisfied. Let ξ take values in $\mathcal{D}(A^\beta)$ a.s. such that $\mathbb{E}\|A^\beta \xi\| < \infty$. If $\alpha \leq \frac{1}{2} - \sigma$ then there exists a unique mild solution of* (1.12) *possessing the regularity:*

$$X \in \mathcal{C}((0,T]; \mathcal{D}(A^\nu)), \quad A^\alpha X \in \mathcal{C}^\gamma([\epsilon, T]; H) \qquad a.s.,$$

and

$$\mathbb{E}\|A^\alpha X\| \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R})$$

for every $0 < \nu < \frac{1}{2}, 0 < \gamma < \sigma$ and $\epsilon \in (0, T]$. Furthermore, the estimate

$$\mathbb{E}\|X(t)\| \leq C[\mathbb{E}\|A^\beta \xi\| + \|A^{-\alpha} F\|_{F^{\beta,\sigma}} t^{\beta - \alpha} + \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))} t^{\beta - \frac{1}{2}}]$$

holds true for every $t \in [0, T]$, where C is some constant depending only on the exponents and constants $\iota_\theta, \upsilon_\theta$ ($\theta \geq 0$).

Remark 3. According to the proof of the above theorems in the next section, we can see that the results in this paper hold true for not only non-random functions $F(t)$ and $G(t)$ but also random ones, i.e. for F and G depending on both $t \in [0, T]$ and $\omega \in \Omega$ and satisfying (H3) and (H4) almost surely. In that case, we shall need more conditions on F and G such as

(i) F and G are measurable with respect to (t, ω).
(ii) $\mathbb{E}\|G\|^2_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))} < \infty$.

The rest of the present paper is organized as follows. In the next section, we shall present the proofs of Theorems 1.2, 1.4 and 1.5. In Section 3, we shall apply the main results to a model arising in neurophysiology.

2. Proof of the main theorems

Proof of Theorem 1.2. Let us show that (1.1) possesses a unique mild solution in the space
$$X \in \mathcal{C}((0, T]; \mathcal{D}(A^{1-\alpha})),$$
which satisfies the estimate (1.11). By (1.5), (1.7) and (H3), we have

$$
\begin{aligned}
\int_0^t \|S(t-s)F(s)\|ds &= \int_0^t \|A^\alpha S(t-s)A^{-\alpha}F(s)\|ds \\
&\leq \iota_\alpha \|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}} \int_0^t (t-s)^{-\alpha}s^{\beta-1}ds \\
&= \iota_\alpha \|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}} B(\beta, 1-\alpha)t^{\beta-\alpha} < \infty, \qquad t \in [0, T].
\end{aligned}
\tag{2.1}
$$

Then the integral $\int_0^t S(t-s)F(s)ds$ is well-defined. Consequently, (1.1) possesses a mild solution given by

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s)ds$$

(the uniqueness is obvious). On the other hand,

$$
\begin{aligned}
&\int_0^t A^{1-\alpha}S(t-s)F(s)ds \\
&= \int_0^t AS(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds + \int_0^t AS(t-s)ds A^{-\alpha}F(t) \\
&= \int_0^t AS(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds + [I - S(t)]A^{-\alpha}F(t).
\end{aligned}
\tag{2.2}
$$

Thanks to (1.5), (1.7) and (H3),

$$\int_0^t \|AS(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]\|ds$$

$$\leq \int_0^t \|AS(t-s)\|\|A^{-\alpha}F(s) - A^{-\alpha}F(t)\|ds$$

$$\leq \iota_1\|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}} \int_0^t (t-s)^{\sigma-1}s^{\beta-\sigma-1}ds$$

$$= \iota_1\|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}} B(\beta-\sigma, \sigma)t^{\beta-1} < \infty, \qquad t \in (0, T].$$

Hence, it is easy to see that $\int_0^t AS(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds$ is continuous in $t \in (0, T]$. In addition, since A is closed, we obtain

$$A \int_0^t S(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds = \int_0^t AS(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds.$$

Therefore, by (2.2), it is seen that

$$A^{1-\alpha} \int_0^t S(t-s)F(s)ds = \int_0^t A^{1-\alpha}S(t-s)F(s)ds$$

and $A^{1-\alpha} \int_0^t S(t-s)F(s)ds$ is continuous in $t \in (0, T]$. As a consequence,

$$A^{1-\alpha}X(t) = A^{1-\alpha}S(t)\xi + A^{1-\alpha} \int_0^t S(t-s)F(s)ds$$

belongs to $\mathcal{C}((0, T]; H)$. Furthermore, by (1.8) and (2.1), X satisfies the estimate (1.11).

Let us next prove that $A^\alpha X(t) = A^\alpha S(t)\xi + A^\alpha \int_0^t S(t-s)F(s)ds$ is continuous in $t \in [0, T]$. Indeed, the first term in the right-hand side of the latter equality is continuous on $[0, T]$, because $\alpha < \beta$ and $A^\alpha S(t)\xi = A^{\alpha-\beta}S(t)A^\beta \xi$. So, it suffices to show that the second term is also continuous on $[0, T]$. Similarly to (2.1), we have

$$\int_0^t \|A^\alpha S(t-s)F(s)\|ds \leq \iota_{2\alpha}\|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}} B(\beta, 1-2\alpha)t^{\beta-2\alpha} < \infty, \ \ t \in [0, T].$$

In addition, since A^α is closed, we obtain

$$A^\alpha \int_0^t S(t-s)F(s)ds = \int_0^t A^\alpha S(t-s)F(s)ds.$$

Hence, it is easy to see that $A^\alpha \int_0^t S(t-s)F(s)ds$ is continuous in $t \in [0, T]$.

Let us now verify that $A^\alpha X \in \mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0, T]; H)$ for every $\gamma \in [0, \sigma]$. We use the expression

$$A^\alpha X(t) = A^\alpha S(t)\xi + \int_0^t A^{2\alpha}S(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds$$

$$+ \int_0^t A^{2\alpha}S(t-s)ds A^{-\alpha}F(t)$$

$$=A^\alpha S(t)\xi + \int_0^t A^{2\alpha} S(t-s)[A^{-\alpha}F(s) - A^{-\alpha}F(t)]ds$$
$$+ A^{2\alpha-1}[I - S(t)]A^{\alpha-1}F(t)$$
$$=J_1(t) + J_2(t) + J_3(t).$$

We will show that J_1, J_2 and J_3 belong to $\mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T];H)$.
Proof for J_1. We prove that

$$J_1 \in \mathcal{F}^{\beta,\gamma}((0,T];H) \subset \mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T];H) \qquad \text{(see (1.6))}.$$

Indeed, applying (1.10) to

$$t^{1-\beta}A^\alpha S(t)\xi = A^{\alpha-1}t^{1-\beta}A^{1-\beta}S(t)A^\beta\xi,$$

we obtain $\lim_{t\to 0} t^{1-\beta}A^\alpha S(t)\xi = 0$. Hence, the condition (1.2) is fulfilled.
On the other hand, by using (1.7), for $0 < s < t \le T$ we have

$$\frac{s^{1-\beta+\gamma}\|A^\alpha[S(t) - S(s)]\xi\|}{(t-s)^\gamma}$$
$$\le \frac{\|A^{\alpha-\gamma-1}[S(t-s) - I]\|}{(t-s)^\gamma} s^{1-\beta+\gamma}\|A^{1-\beta+\gamma}S(s)A^\beta\xi\|$$
$$\le \frac{\|\int_0^{t-s} A^{\alpha-\gamma}S(u)du\|}{(t-s)^\gamma} \iota_{1-\beta+\gamma}\|A^\beta\xi\|. \qquad (2.3)$$

If $\alpha \ge \gamma$ then

$$\frac{s^{1-\beta+\gamma}\|A^\alpha[S(t) - S(s)]\xi\|}{(t-s)^\gamma} \le \frac{\int_0^{t-s} \iota_{\alpha-\gamma}u^{\gamma-\alpha}du}{(t-s)^\gamma} \iota_{1-\beta+\gamma}\|A^\beta\xi\|$$
$$= \frac{\iota_{\alpha-\gamma}\iota_{1-\beta+\gamma}\|A^\beta\xi\|}{1+\gamma-\alpha}(t-s)^{1-\alpha}. \qquad (2.4)$$

If $\alpha < \gamma$ then by (1.8) and (1.9)

$$\frac{s^{1-\beta+\gamma}\|A^\alpha[S(t) - S(s)]\xi\|}{(t-s)^\gamma} \le \frac{\int_0^{t-s} \upsilon_{\gamma-\alpha}\iota_0 du}{(t-s)^\gamma} \iota_{1-\beta+\gamma}\|A^\beta\xi\|$$
$$= \upsilon_{\gamma-\alpha}\iota_0\iota_{1-\beta+\gamma}\|A^\beta\xi\|(t-s)^{1-\gamma}. \qquad (2.5)$$

Hence,

$$\sup_{0 \le s < t \le T} \frac{s^{1-\beta+\gamma}\|A^\alpha[S(t) - S(s)]\xi\|}{(t-s)^\gamma} < \infty$$

and

$$\lim_{t\to 0} \sup_{0 < s < t} \frac{s^{1-\beta+\gamma}\|A^\alpha[S(t) - S(s)]\xi\|}{(t-s)^\gamma} = 0.$$

Therefore, the conditions (1.3) and (1.4) are also satisfied. We have thus verified that $J_1 \in \mathcal{F}^{\beta,\gamma}((0,T];H)$.

Proof for J_2. Using (1.5) and (H3) we have

$$
\begin{aligned}
\|J_2(t)\| &\leq \int_0^t \|A^{2\alpha} S(t-s)\| \|A^{-\alpha} F(s) - A^{-\alpha} F(t)\| ds \\
&\leq \iota_{2\alpha} w_{A^{-\alpha}F}(t) \int_0^t (t-s)^{\sigma-2\alpha} s^{\beta-\sigma-1} ds \\
&= \iota_{2\alpha} w_{A^{-\alpha}F}(t) B(\beta-\sigma, 1+\sigma-2\alpha) t^{\beta-2\alpha},
\end{aligned}
$$

here

$$
w_{A^{-\alpha}F}(t) = \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma} \|A^{-\alpha} F(t) - A^{-\alpha} F(s)\|}{(t-s)^\sigma} \qquad \text{(see (1.4))}.
$$

This implies that

$$
\lim_{t \to 0} t^{1-(\beta-\sigma+\gamma)} J_2(t) = 0 \qquad \text{in } H. \tag{2.6}
$$

We next observe that for $0 < s < t \leq T$

$$
\begin{aligned}
J_2(t) - J_2(s) &= \int_s^t A^{2\alpha} S(t-u) [A^{-\alpha} F(u) - A^{-\alpha} F(t)] du \\
&\quad + [S(t-s) - I] \int_0^s A^{2\alpha} S(s-u) [A^{-\alpha} F(u) - A^{-\alpha} F(s)] du \\
&\quad + \int_0^s A^{2\alpha} S(t-u) [A^{-\alpha} F(s) - A^{-\alpha} F(t)] du \\
&= J_{21}(t,s) + J_{22}(t,s) + J_{23}(t,s). \tag{2.7}
\end{aligned}
$$

For $J_{21}(t,s)$, by (1.5) and (1.7) we have the estimate

$$
\begin{aligned}
\|J_{21}(t,s)\| &\leq \int_s^t \|A^{2\alpha} S(t-u)\| \|A^{-\alpha} F(u) - A^{-\alpha} F(t)\| du \\
&\leq \int_s^t \iota_{2\alpha} w_{A^{-\alpha}F}(t)(t-u)^{\sigma-2\alpha} u^{\beta-\sigma-1} du \\
&\leq \iota_{2\alpha} w_{A^{-\alpha}F}(t) s^{\beta-\sigma-1} \int_s^t (t-u)^{\sigma-2\alpha} du \\
&= \frac{\iota_{2\alpha} w_{A^{-\alpha}F}(t) s^{(\beta-\sigma+\gamma)-\gamma-1} (t-s)^{1+\sigma-2\alpha}}{1+\sigma-2\alpha} \\
&\leq C_1 w_{A^{-\alpha}F}(t) s^{(\beta-\sigma+\gamma)-\gamma-1} (t-s)^\gamma, \tag{2.8}
\end{aligned}
$$

where $C_1 > 0$ is some positive constant.

For $J_{22}(t,s)$, we estimate its norm as follows.

$$
\|J_{22}(t,s)\|
$$

$$= \left\| \int_0^{t-s} AS(r)dr \int_0^s A^{2\alpha} S(s-u)[A^{-\alpha}F(u) - A^{-\alpha}F(s)]du \right\|$$

$$= \left\| \int_0^{t-s} \int_0^s A^{1+2\alpha} S(r+s-u)[A^{-\alpha}F(u) - A^{-\alpha}F(s)]dudr \right\|$$

$$\leq \iota_{1+2\alpha} w_{A^{-\alpha}F}(s) \int_0^{t-s} \int_0^s (r+s-u)^{-1-2\alpha}(s-u)^\sigma u^{\beta-\sigma-1} dudr$$

$$= \frac{\iota_{1+2\alpha} w_{A^{-\alpha}F}(s)}{2\alpha} \int_0^s [(s-u)^{-2\alpha} - (t-u)^{-2\alpha}](s-u)^\sigma u^{\beta-\sigma-1} du$$

$$= \frac{\iota_{1+2\alpha} w_{A^{-\alpha}F}(s)}{2\alpha} \int_0^s [(t-u)^{2\alpha} - (s-u)^{2\alpha}](t-u)^{-2\alpha}(s-u)^{\sigma-2\alpha}$$
$$\times u^{\beta-\sigma-1} du$$

$$\leq \frac{\iota_{1+2\alpha}(t-s)^{2\alpha} w_{A^{-\alpha}F}(s)}{2\alpha} \int_0^s (t-u)^{-2\alpha}(s-u)^{\sigma-2\alpha} u^{\beta-\sigma-1} du$$

$$= \frac{\iota_{1+2\alpha} w_{A^{-\alpha}F}(s)}{2\alpha}(t-s)^{2\alpha} \int_0^s (t-s+u)^{-2\alpha} u^{\sigma-2\alpha}(s-u)^{\beta-\sigma-1} du, \quad (2.9)$$

here we used the inequality $(t-u)^{2\alpha} - (s-u)^{2\alpha} \leq (t-s)^{2\alpha}$. We have

$$(t-s)^{2\alpha} \int_{\frac{s}{2}}^s (t-s+u)^{-2\alpha} u^{\sigma-2\alpha}(s-u)^{\beta-\sigma-1} du$$

$$=(t-s)^\gamma \int_{\frac{s}{2}}^s (t-s)^{2\alpha-\gamma}(t-s+u)^{-2\alpha} u^{1+\sigma-2\alpha} u^{-1}(s-u)^{\beta-\sigma-1} du$$

$$\leq 2(t-s)^{\gamma} s^{-1} \int_{\frac{s}{2}}^s [(t-s)^{2\alpha-\gamma}(t-s+u)^{-2\alpha} u^{1+\sigma-2\alpha}](s-u)^{\beta-\sigma-1} du.$$

Since there exists $C_2 > 0$ such that for every $\frac{s}{2} \leq u \leq s$

$$(t-s)^{2\alpha-\gamma}(t-s+u)^{-2\alpha} u^{1+\sigma-2\alpha}$$
$$= \left(\frac{t-s}{t-s+u}\right)^{2\alpha-\gamma} \left(\frac{u}{t-s+u}\right)^\gamma u^{1+\sigma-2\alpha-\gamma} \leq C_2,$$

we obtain

$$(t-s)^{2\alpha} \int_{\frac{s}{2}}^s (t-s+u)^{-2\alpha} u^{\sigma-2\alpha}(s-u)^{\beta-\sigma-1} du$$

$$\leq 2C_2(t-s)^\gamma s^{-1} \int_0^s (s-u)^{\beta-\sigma-1} du$$

$$= \frac{2C_2(t-s)^\gamma s^{(\beta-\sigma+\gamma)-\gamma-1}}{\beta-\sigma}. \quad (2.10)$$

Meanwhile,

$$(t-s)^{2\alpha} \int_0^{\frac{s}{2}} (t-s+u)^{-2\alpha} u^{\sigma-2\alpha}(s-u)^{\beta-\sigma-1} du$$

$$\leq 2^{1-\beta+\sigma} s^{\beta-\sigma-1}(t-s)^{2\alpha} \int_0^{\frac{s}{2}} (t-s+u)^{-2\alpha} u^{\sigma-2\alpha} du$$

$$\leq 2^{1-\beta+\sigma} s^{\beta-\sigma-1}(t-s)^{1+\sigma-2\alpha} \int_0^\infty (1+r)^{-2\alpha} r^{\sigma-2\alpha} dr$$

$$= 2^{1-\beta+\sigma}(t-s)^{1+\sigma-2\alpha-\gamma} \int_0^\infty (1+r)^{-2\alpha} r^{\sigma-2\alpha} dr s^{\beta-\sigma-1}(t-s)^\gamma$$

$$\leq 2^{1-\beta+\sigma} T^{1+\sigma-2\alpha-\gamma} \int_0^\infty (1+r)^{-2\alpha} r^{\sigma-2\alpha} dr s^{\beta-\sigma-1}(t-s)^\gamma$$

$$\leq C_3 s^{\beta-\sigma-1}(t-s)^\gamma, \tag{2.11}$$

where

$$C_3 = 2^{1-\beta+\sigma} T^{1+\sigma-2\alpha-\gamma} \int_0^\infty (1+r)^{-2\alpha} r^{\sigma-2\alpha} dr < \infty \quad (\text{since } 1+\sigma-4\alpha < 0).$$

Taking the sum of (2.10) and (2.11) and substituting it for the integral in the right-hand side of (2.9), we gain

$$\|J_{22}(t,s)\| \leq \left(\frac{2C_2}{\beta-\sigma} + C_3 \right) \frac{\iota_{1+2\alpha}}{2\alpha} w_{A^{-\alpha}F}(s) s^{(\beta-\sigma+\gamma)-\gamma-1}(t-s)^\gamma. \tag{2.12}$$

For $J_{23}(t,s)$, by using (1.5)

$$\|J_{23}(t,s)\| = \|A^{2\alpha-1}[S(t-s)-S(t)][A^{-\alpha}F(s)-A^{-\alpha}F(t)]\|$$
$$\leq \|A^{2\alpha-1}[S(t-s)-S(t)]\|(t-s)^{\sigma-\gamma} w_{A^{-\alpha}F}(t) s^{\beta-\sigma-1}(t-s)^\gamma.$$

Then it is clear that there exists $C_4 > 0$ such that

$$\|J_{23}(t,s)\| \leq C_4 w_{A^{-\alpha}F}(t) s^{(\beta-\sigma+\gamma)-\gamma-1}(t-s)^\gamma. \tag{2.13}$$

Thanks to (2.6), (2.7), (2.8), (2.12) and (2.13), we conclude that

$$J_2 \in \mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T];H).$$

Proof for J_3. Since $t^{1-\beta} A^{-\alpha} F(t)$ has a limit as $t \to 0$,

$$\lim_{t\to 0} t^{1-\beta} J_3(t) = \lim_{t\to 0} A^{2\alpha-1}[I-S(t)] t^{1-\beta} A^{-\alpha} F(t) = 0. \tag{2.14}$$

We next write

$$J_3(t) - J_3(s) = A^{2\alpha-1}[I-S(t)][A^{-\alpha}F(t)-A^{-\alpha}F(s)]$$
$$+ A^{2\alpha-1}[I-S(t-s)]S(s)A^{-\alpha}F(s). \tag{2.15}$$

Let us give estimate for the norm of the first term in the right-hand side of the latter equality. Due to (1.5), (1.8) and (1.9) there exists $C_5 > 0$ such that

$$\|A^{2\alpha-1}[I-S(t)][A^{-\alpha}F(t)-A^{-\alpha}F(s)]\|$$

$$\leq \|A^{2\alpha-1}[I - S(t)]\|w_{A^{-\alpha}F}(t)s^{\beta-\sigma-1}(t-s)^{\sigma}$$
$$\leq C_5 w_{A^{-\alpha}F}(t)s^{(\beta-\sigma+\gamma)-\gamma-1}(t-s)^{\gamma}. \tag{2.16}$$

For the norm of the second term, we have

$$\|A^{2\alpha-1}[S(t-s) - I]S(s)A^{-\alpha}F(s)\|$$
$$\leq \|[S(t-s) - I]A^{2\alpha-\sigma-1}\|s^{\beta-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|$$
$$\leq \Big\|\int_0^{t-s} A^{2\alpha-\sigma}S(r)dr\Big\|s^{\beta-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|.$$

If $2\alpha \geq \sigma$ then

$$\|A^{2\alpha-1}[S(t-s) - I]S(s)A^{-\alpha}F(s)\|$$
$$\leq \int_0^{t-s} \iota_{2\alpha-\sigma}r^{\sigma-2\alpha}dr s^{\beta-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|$$
$$= \frac{\iota_{2\alpha-\sigma}}{1-2\alpha+\sigma}(t-s)^{1-2\alpha+\sigma-\gamma}(t-s)^{\gamma}s^{\beta-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|$$
$$\leq \frac{\iota_{2\alpha-\sigma}T^{1-2\alpha+\sigma-\gamma}}{1-2\alpha+\sigma}(t-s)^{\gamma}s^{(\beta-\sigma+\gamma)-\gamma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|.$$

Meanwhile, if $2\alpha < \sigma$ then by using (1.7) and (1.9)

$$\|A^{2\alpha-1}[S(t-s) - I]S(s)A^{-\alpha}F(s)\|$$
$$\leq \int_0^{t-s} \upsilon_{\sigma-2\alpha}\iota_0 dr s^{\beta-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|$$
$$= \upsilon_{\sigma-2\alpha}\iota_0(t-s)^{1-\gamma}(t-s)^{\gamma}s^{(\beta-\sigma+\gamma)-\gamma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|$$
$$\leq \upsilon_{\sigma-2\alpha}\iota_0 T^{1-\gamma}(t-s)^{\gamma}s^{(\beta-\sigma+\gamma)-\gamma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|.$$

Hence, there exists $C_6 > 0$ such that

$$\|A^{2\alpha-1}[S(t-s) - I]S(s)A^{-\alpha}F(s)\|$$
$$\leq C_6(t-s)^{\gamma}s^{(\beta-\sigma+\gamma)-\gamma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\|. \tag{2.17}$$

On the other hand, since $s^{1-\beta}A^{-\alpha}F(s)$ has a limit as $s \to 0$, by (1.10)

$$\lim_{s\to 0} \|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha}F(s)\| = 0.$$

According to (2.14), (2.15), (2.16) and (2.17), J_3 is then verified to be in $\mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T];H)$.

We have thus proved that $A^{\alpha}X$ belongs to $\mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T];H)$ for every $0 \leq \gamma \leq \sigma$. It completes the proof of the theorem. $\qquad\square$

Proof of Theorem 1.4. Throughout this proof, the notation C will stand for a universal constant which is determined in each occurrence by the exponents and by constants $\iota_{\theta}, \upsilon_{\theta}$ ($\theta \geq 0$) in a specific way. We will use the following property of stochastic integrals (see [5] for the proof).

Lemma 2.1. *Let ϕ be a measurable function from $([0,T], \mathcal{B}([0,T]))$ to $(L_2(U;H),$ $\mathcal{B}(L_2(U;H)))$ satisfying the condition*

$$\int_0^t \|\phi(s)\|_{L_2(U;H)}^2 ds < \infty, \qquad\qquad t \in [0,T].$$

Then the stochastic integral $\int_0^t \phi(s)dW(s)$ is well-defined and is continuous in $t \in [0,T]$. Furthermore,

$$\mathbb{E}\Big\| \int_0^t \phi(s)dW(s)\Big\|^2 = \int_0^t \|\phi(s)\|_{L_2(U;H)}^2 ds.$$

In fact, the integral is a continuous square integrable martingale on $[0,T]$. Here and then, the continuity means that almost surely trajectories of the stochastic process are continuous in time.

We divide the proof into several steps.

Step 1. Let us observe that there exists a unique continuous mild solution of (1.12), which satisfies the estimate (1.13). Indeed, due to (1.5) and (1.8), we have

$$
\begin{aligned}
&\int_0^t \|S(t-s)G(s)\|_{L_2(U;H)}^2 ds \\
&\leq \int_0^t \|S(t-s)\|^2 \|G(s)\|_{L_2(U;H)}^2 ds \\
&\leq \int_0^t \iota_0^2 \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}^2 s^{2(\beta-1)} ds \\
&= \frac{\iota_0^2 \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}^2 t^{2\beta-1}}{2\beta-1} < \infty, \qquad\qquad t \in [0,T]. \qquad (2.18)
\end{aligned}
$$

Therefore, by using Lemma 2.1, it is easy to see that the stochastic convolution

$$W_G(t) = \int_0^t S(t-s)G(s)dW(s)$$

is continuous in $t \in [0,T]$. Thus, $X(t) = S(t)\xi + W_G(t)$ is a unique continuous mild solution of (1.12). Furthermore, by (1.8), (1.9) and (2.18),

$$
\begin{aligned}
\mathbb{E}\|X(t)\| &\leq \mathbb{E}\|S(t)\xi\| + \mathbb{E}\|W_G(t)\| \\
&\leq \mathbb{E}\|S(t)\xi\| + \sqrt{\mathbb{E}\|W_G(t)\|^2} \\
&= \mathbb{E}\|A^{-\beta}S(t)A^\beta\xi\| + \sqrt{\int_0^t \|S(t-s)G(s)\|_{L_2(U;H)}^2 ds} \\
&\leq C[\mathbb{E}\|A^\beta\xi\| + \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))} t^{\beta-\frac{1}{2}}], \qquad\qquad t \in [0,T].
\end{aligned}
$$

Step 2. Let us show that for every $\nu \in (0, \frac{1}{2})$, $X \in \mathcal{C}((0, T]; \mathcal{D}(A^\nu))$ a.s.. Indeed, we have

$$\int_0^t \|A^\nu S(t-s)G(s)\|_{L_2(U;H)}^2 ds$$

$$\leq \int_0^t \|A^\nu S(t-s)\|^2 \|G(s)\|_{L_2(U;H)}^2 ds$$

$$\leq \int_0^t \iota_\nu^2 \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}^2 (t-s)^{-2\nu} s^{2(\beta-1)} ds$$

$$= \iota_\nu^2 \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}^2 B(2\beta-1, 1-2\nu) t^{2(\beta-\nu)-1}$$

$$< \infty, \qquad\qquad t \in (0, T].$$

Lemma 2.1 then provides that the stochastic integral $\int_0^t A^\nu S(t-s)G(s)dW(s)$ is continuous on $(0, T]$. Since A^ν is closed, we obtain

$$A^\nu W_G(t) = \int_0^t A^\nu S(t-s)G(s)dW(s).$$

Hence $A^\nu W_G$ is continuous on $(0, T]$. On the other hand, since $A^\nu S(t)\xi = A^{\nu-\beta}S(t)A^\beta \xi$, $A^\nu S(\cdot)\xi$ is continuous on $[0, T]$. In this way, we conclude that $A^\nu X(t) = A^\nu S(t)\xi + A^\nu W_G(t)$ is continuous in $t \in (0, T]$, i.e. for every $\nu \in (0, \frac{1}{2})$, $X \in \mathcal{C}((0, T]; \mathcal{D}(A^\nu))$ a.s..

Step 3. Let us verify that for every $\alpha_1 \in (0, \frac{1}{2} - \sigma]$

$$\mathbb{E}\|A^{\alpha_1}W_G(t) - A^{\alpha_1}W_G(s)\|^2 \leq m(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}, \qquad 0 < s \leq t \leq T$$

where $m(t)$ is some increasing function on $(0, T]$ such that $\lim_{t \to 0} m(t) = 0$.

From the expression

$$A^{\alpha_1}W_G(t) = \int_0^t A^{\alpha_1}S(t-r)[G(r) - G(t)]dW(r) + \int_0^t A^{\alpha_1}S(t-r)G(t)dW(r),$$

we have

$$A^{\alpha_1}W_G(t) - A^{\alpha_1}W_G(s)$$

$$= \int_s^t A^{\alpha_1}S(t-r)[G(r) - G(t)]dW(r)$$

$$+ \int_0^s A^{\alpha_1}S(t-r)[G(r) - G(t)]dW(r)$$

$$- \int_0^s A^{\alpha_1}S(s-r)[G(r) - G(s)]dW(r)$$

$$+ \int_s^t A^{\alpha_1}S(t-r)G(t)dW(r) + \int_0^s A^{\alpha_1}S(t-r)G(t)dW(r)$$

$$- \int_0^s A^{\alpha_1}S(s-r)G(s)dW(r)$$

$$= \int_s^t A^{\alpha_1} S(t-r)[G(r)-G(t)]dW(r)$$

$$+ \int_0^s A^{\alpha_1} S(t-s)S(s-r)[G(r)-G(s)+G(s)-G(t)]dW(r)$$

$$- \int_0^s A^{\alpha_1} S(s-r)[G(r)-G(s)]dW(r) + \int_s^t A^{\alpha_1} S(t-r)G(t)dW(r)$$

$$+ \int_0^s [A^{\alpha_1} S(t-r)G(t) - A^{\alpha_1} S(s-r)G(s)]dW(r)$$

$$= \int_s^t A^{\alpha_1} S(t-r)[G(r)-G(t)]dW(r)$$

$$+ \int_0^s [S(t-s)-I]A^{\alpha_1} S(s-r)[G(r)-G(s)]dW(r)$$

$$+ \int_0^s A^{\alpha_1} S(t-r)[G(s)-G(t)]dW(r) + \int_s^t A^{\alpha_1} S(t-r)G(t)dW(r)$$

$$+ \int_0^s A^{\alpha_1} S(s-r)[G(t)-G(s)]dW(r)$$

$$+ \int_0^s A^{\alpha_1}[S(t-r)-S(s-r)]G(t)dW(r)$$

$$= K_1 + K_2 + K_3 + K_4 + K_5 + K_6.$$

Let us give estimates for the expectation of the square of norm of K_i ($i = 1, \ldots, 6$). For K_1, thanks to (1.5) and (1.7) we have

$$\mathbb{E}\|K_1\|^2 = \int_s^t \|A^{\alpha_1} S(t-r)[G(r)-G(t)]\|_{L_2(U;H)}^2 dr$$

$$\leq \int_s^t \|A^{\alpha_1} S(t-r)\|^2 \|G(r)-G(t)\|_{L_2(U;H)}^2 dr$$

$$\leq \iota_{\alpha_1}^2 w_G(t)^2 \int_s^t (t-r)^{2(\sigma-\alpha_1)} r^{2(\beta-\sigma-1)} dr$$

$$\leq \iota_{\alpha_1}^2 w_G(t)^2 s^{2(\beta-\sigma-1)} \int_s^t (t-r)^{2(\sigma-\alpha_1)} dr$$

$$= \iota_{\alpha_1}^2 w_G(t)^2 s^{2(\beta-\sigma-1)} \frac{(t-s)^{1+2(\sigma-\alpha_1)}}{1+2(\sigma-\alpha_1)}$$

$$\leq C w_G(t)^2 s^{2(\beta-\sigma-1)} (t-s)^{2\sigma},$$

where

$$w_G(t) = \sup_{0 \leq s \leq t} \frac{s^{1-\beta+\sigma}\|G(t)-G(s)\|_{L_2(U;H)}}{(t-s)^\sigma}.$$

For K_2 we have

$$\mathbb{E}\|K_2\|^2 = \int_0^s \left\| \int_0^{t-s} AS(\rho)d\rho A^{\alpha_1} S(s-r)[G(r)-G(s)] \right\|_{L_2(U;H)}^2 dr$$

$$\leq \int_0^s \left\| \int_0^{t-s} A^{1-\sigma} S(\rho) d\rho \right\|^2 \|A^{\alpha_1+\sigma} S(s-r)\|^2$$
$$\times \|G(r) - G(s)\|_{L_2(U;H)}^2 dr$$
$$\leq \iota_{1-\sigma}^2 \iota_{\alpha_1+\sigma}^2 w_G(s)^2$$
$$\times \int_0^s \left(\int_0^{t-s} \rho^{\sigma-1} d\rho \right)^2 (s-r)^{-2(\alpha_1+\sigma)} (s-r)^{2\sigma} r^{2(\beta-\sigma-1)} dr$$
$$\leq C w_G(s)^2 \int_0^s (t-s)^{2\sigma} (s-r)^{-2\alpha_1} r^{2(\beta-\sigma-1)} dr$$
$$= CB(2\beta - 2\sigma - 1, 1 - 2\alpha_1) s^{2(\beta-\sigma-\alpha_1)-1} w_G(s)^2 (t-s)^{2\sigma}$$
$$\leq C w_G(s)^2 s^{2(\beta-\sigma-1)} (t-s)^{2\sigma}.$$

For K_3,

$$\mathbb{E}\|K_3\|^2 = \int_0^s \|A^{\alpha_1} S(t-r)[G(s) - G(t)]\|_{L_2(U;H)}^2 dr$$
$$\leq \int_0^s \|A^{\alpha_1} S(t-r)\|^2 \|G(s) - G(t)\|_{L_2(U;H)}^2 dr$$
$$\leq C \int_0^s (t-r)^{-2\alpha_1} dr w_G(t)^2 (t-s)^{2\sigma} s^{2(\beta-\sigma-1)}$$
$$\leq C[t^{1-2\alpha_1} - (t-s)^{1-2\alpha_1}] w_G(t)^2 (t-s)^{2\sigma} s^{2(\beta-\sigma-1)}$$
$$\leq C w_G(t)^2 s^{2(\beta-\sigma-1)} (t-s)^{2\sigma}.$$

For K_4,

$$\mathbb{E}\|K_4\|^2 = \int_s^t \|A_1^{\alpha} S(t-r) G(t)\|_{L_2(U;H)}^2 dr$$
$$\leq \int_s^t \|A^{\alpha_1} S(t-r)\|^2 \|G(t)\|_{L_2(U;H)}^2 dr$$
$$\leq C \int_s^t (t-r)^{-2\alpha_1} dr \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}^2 t^{2(\beta-1)}$$
$$\leq C t^{2(\beta-1)} (t-s)^{1-2\alpha_1}$$
$$= C t^{2\sigma} (t-s)^{1-2\alpha_1-2\sigma} t^{2(\beta-\sigma-1)} (t-s)^{2\sigma}$$
$$\leq C t^{2\sigma} T^{1-2\alpha_1-2\sigma} s^{2(\beta-\sigma-1)} (t-s)^{2\sigma}$$
$$\leq C t^{2\sigma} s^{2(\beta-\sigma-1)} (t-s)^{2\sigma}.$$

For K_5,

$$\mathbb{E}\|K_5\|^2 = \int_0^s \|A^{\alpha_1} S(s-r)[G(t) - G(s)]\|_{L_2(U;H)}^2 dr$$
$$\leq \int_0^s \|A^{\alpha_1} S(s-r)\|^2 \|G(t) - G(s)\|^2 dr$$

$$\leq C \int_0^s (s-r)^{-2\alpha_1} dr w_G(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}$$
$$\leq C s^{2(\beta-\sigma-\alpha_1)-1} w_G(t)^2 (t-s)^{2\sigma}$$
$$\leq C w_G(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.$$

Finally, for K_6 we have

$$\mathbb{E}\|K_6\|^2 = \int_0^s \|A^{\alpha_1}[S(t-r) - S(s-r)]G(t)\|^2_{L_2(U;H)} dr$$
$$\leq \int_0^s \|A^{\alpha_1+\sigma} S(s-r)\|^2 \|S(t-s) - I]A^{-\sigma}\|^2 \|G(t)\|^2_{L_2(U;H)} dr$$
$$= \int_0^s \|A^{\alpha_1+\sigma} S(s-r)\|^2 \Big\| \int_0^{t-s} A^{1-\sigma} S(\rho) d\rho \Big\|^2 dr \|G(t)\|^2_{L_2(U;H)}$$
$$\leq \iota^2_{\alpha_1+\sigma} \iota^2_{1-\sigma} \int_0^s (s-r)^{-2(\alpha_1+\sigma)} dr \Big[\int_0^{t-s} \rho^{\sigma-1} d\rho \Big]^2$$
$$\times \|G\|^2_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))} t^{2(\beta-1)}$$
$$\leq C s^{1-2(\alpha_1+\sigma)} t^{2\sigma} t^{2(\beta-\sigma-1)}(t-s)^{2\sigma}$$
$$\leq C T^{1-2(\alpha_1+\sigma)} t^{2\sigma} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}$$
$$\leq C t^{2\sigma} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.$$

In this way, we conclude that

$$\mathbb{E}\|A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)\|^2 \leq 6 \sum_{i=1}^{6} \mathbb{E}\|K_i\|^2$$
$$\leq m(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma},$$

where $m(\cdot)$ is some increasing function on $(0,T]$ such that $\lim_{t\to 0} m(t) = 0$.

Step 4. Let us verify that for $0 < \gamma < \sigma$, $0 < \alpha_1 \leq \frac{1}{2} - \sigma$ and $\epsilon \in (0,T]$

$$A^{\alpha_1} X \in \mathcal{C}^{\gamma}([\epsilon, T]; H) \qquad \text{a.s.,}$$

and

$$\mathbb{E}\|A^{\alpha_1} X\| \in \mathcal{F}^{\beta,\sigma}((0,T];\mathbb{R}).$$

On account of **Step 2**, $A^{\alpha_1} W_G$ is a Gaussian process. Using the estimate in **Step 3** we apply the Kolmogorov continuity theorem to the process $A^{\alpha_1} W_G$. We then obtain

$$A^{\alpha_1} W_G \in \mathcal{C}^{\gamma}([\epsilon, T]; H).$$

Furthermore, it has already verified in Theorem 1.2 that

$$J_1 = A^{\alpha_1} S(t)\xi \in \mathcal{F}^{\beta,\gamma}((0,T]; H)$$

(this result holds true for every $\alpha_1 \in (0,1)$). With a remark that

$$\mathcal{F}^{\beta,\gamma}((0,T]; H) \subset \mathcal{C}^{\gamma}([\epsilon, T]; H), \qquad 0 < \epsilon \leq T,$$

we arrive at the first statement, i.e.

$$A^{\alpha_1} X = J_1 + A^{\alpha_1} W_G \in \mathcal{C}^\gamma([\epsilon, T]; H).$$

We shall show the second one. Due to the estimate in **Step 3**,

$$[\mathbb{E}\|A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)\|]^2 \leq \mathbb{E}\|A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)\|^2$$
$$\leq m(t)^2 s^{2(\beta - \sigma - 1)} (t - s)^{2\sigma}.$$

Hence,

$$\frac{s^{1-\beta+\sigma} \mathbb{E}\|A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)\|}{(t-s)^\sigma} \leq m(t). \qquad (2.19)$$

Furthermore, by choosing $\gamma = \sigma$ in the estimates (2.3), (2.4) and (2.5), which hold true for every $\alpha_1 \in (0, 1)$, we obtain

$$\frac{s^{1-\beta+\sigma} \|A^{\alpha_1}[S(t) - S(s)]\xi\|}{(t-s)^\sigma} \leq C\|A^\beta \xi\| \max\{(t-s)^{1-\alpha_1}, (t-s)^{1-\sigma}\}.$$

Taking expectation of both the hand sides of the above estimate, it follows that

$$\frac{s^{1-\beta+\sigma} \mathbb{E}\|J_1(t) - J_1(s)\|}{(t-s)^\sigma} \leq C\mathbb{E}\|A^\beta \xi\| \max\{t^{1-\alpha_1}, t^{1-\sigma}\}. \qquad (2.20)$$

Combining (2.19) and (2.20) yields that

$$\frac{s^{1-\beta+\sigma} |\mathbb{E}\|A^{\alpha_1} X(t)\| - \mathbb{E}\|A^{\alpha_1} X(s)\||}{(t-s)^\sigma}$$
$$\leq \frac{s^{1-\beta+\sigma} \mathbb{E}\|A^{\alpha_1} X(t) - A^{\alpha_1} X(s)\|}{(t-s)^\sigma}$$
$$= \frac{s^{1-\beta+\sigma} \mathbb{E}\|[A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)] + [J_1(t) - J_1(s)]\|}{(t-s)^\sigma}$$
$$\leq \frac{s^{1-\beta+\sigma} [\mathbb{E}\|A^{\alpha_1} W_G(t) - A^{\alpha_1} W_G(s)\| + \mathbb{E}\|J_1(t) - J_1(s)\|]}{(t-s)^\sigma}$$
$$\leq m(t) + C\mathbb{E}\|A^\beta \xi\| \max\{t^{1-\alpha_1}, t^{1-\sigma}\}.$$

Therefore,

$$\sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma} |\mathbb{E}\|A^{\alpha_1} X(t)\| - \mathbb{E}\|A^{\alpha_1} X(s)\||}{(t-s)^\sigma} < \infty \qquad (2.21)$$

and

$$\lim_{t \to 0} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma} |\mathbb{E}\|A^{\alpha_1} X(t)\| - \mathbb{E}\|A^{\alpha_1} X(s)\||}{(t-s)^\sigma} = 0. \qquad (2.22)$$

On the other hand, by using the equality

$$\mathbb{E}\|A^{\alpha_1} W_G(t)\|^2 = \int_0^t \|A^{\alpha_1} S(t-s) G(s)\|_{L_2(U;H)}^2 ds$$

and the estimate in **Step 2**, we have

$$t^{1-\beta}\mathbb{E}\|A^{\alpha_1}W_G(t)\| \leq t^{1-\beta}\sqrt{\mathbb{E}\|A^{\alpha_1}W_G(t)\|^2}$$
$$\leq \iota_{\alpha_1}\|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}t^{\frac{1}{2}-\alpha_1}\sqrt{B(2\beta-1,1-2\alpha_1)}.$$

Hence,

$$\lim_{t\to 0}t^{1-\beta}\mathbb{E}\|A^{\alpha_1}W_G(t)\| = 0.$$

In addition, thanks to (1.10),

$$\lim_{t\to 0}t^{1-\beta}\mathbb{E}\|J(t)\| = \lim_{t\to 0}\mathbb{E}\|A^{\alpha_1-1}t^{1-\beta}A^{1-\beta}S(t)A^{\beta}\xi\| = 0.$$

In this way, we thus proved that

$$\lim_{t\to 0}t^{1-\beta}\mathbb{E}\|A^{\alpha_1}X(t)\| = \lim_{t\to 0}t^{1-\beta}\mathbb{E}\|J(t) + A^{\alpha_1}W_G(t)\| = 0. \qquad (2.23)$$

By (2.21), (2.22) and (2.23), the second statement has been verified. We complete the proof of the theorem. $\qquad \square$

Proof of Theorem 1.5. Clearly, the assumptions of Theorem 1.2 and Theorem 1.4 are fulfilled. Therefore, it is easy to see that Theorem 1.5 follows from these two theorems. $\qquad \square$

3. An application to neurophysiology

In this section, we will apply our results to a model arising in neurophysiology, which has been proposed by Walsh [18]. Nerve cells operate by a mixture of chemical, biological and electrical properties. We shall regard them as long thin cylinders, which act much like electrical cables. Denote by $V(t,x)$ the electrical potential at time t and the point x. If we identify such a cylinder with the interval $[0, L]$ then V satisfies a nonlinear equations coupled with a system of ordinary differential equations, called the Hodgkin-Huxley equations. In some certain ranges of the values of the potential, the equations are approximated by the cable equation:

$$\begin{cases} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V, \\ V\colon [0,\infty) \times [0,L] \to \mathbb{R}, \\ t \geq 0, x \in [0,L]. \end{cases} \qquad (3.1)$$

Let us consider (3.1) with an additional external force, called impulses of current. Neurons receive these impulses of current via synapses on their surface. Denote by $F(t)$ the current arriving at time t, then the electrical potential satisfies the equation:

$$\begin{cases} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V + F(t), \\ V\colon [0,\infty) \times [0,L] \to \mathbb{R}, \\ t \geq 0, x \in [0,L]. \end{cases}$$

Suppose now that the current F is perturbed by a space-time white noise $\dot{W}$, i.e.

$$F(t) \rightsquigarrow F(t) + G(t)\dot{W}.$$

This leads us to the stochastic partial differential equation:

$$
\begin{cases}
\dfrac{\partial V}{\partial t}(t,x) = \dfrac{\partial^2 V}{\partial x^2}(t,x) - V(t,x) + F(t) + G(t)\dfrac{\partial W(t)}{\partial t}, & t > 0, x \in [0, L], \\[2mm]
\dfrac{\partial V}{\partial x}(0,t) = \dfrac{\partial V}{\partial x}(L,t) = 0, & t > 0, \\[2mm]
V(0,x) = V_0(x), & x \in (0, L),
\end{cases}
\tag{3.2}
$$

where W is a cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, V_0 is an $\mathcal{F}_0$-measurable random variable. More explanations for the construction of the system (3.2) can be found in [18].

Let us next construct an abstract formulation for (3.2). We will handle the equation in the Hilbert space $H = L_2([0, L])$. We assume that the cylindrical Wiener process W takes value on a separable Hilbert space U, and that F and G are measurable from $([0, \infty), \mathcal{B}([0, \infty)))$ to $(H, \mathcal{B}(H))$ and $(L_2(U; H), \mathcal{B}(L_2(U; H)),$ respectively. Denote by A the realization of the operator $-\frac{\partial^2}{\partial x^2} + I$ in H under the Neumann boundary conditions on the boundary $\{0, L\}$. We formulate the equation (3.2) as the Cauchy problem for an abstract stochastic linear equation

$$
\begin{cases}
dV + AV = F(t) + G(t)dW(t), & 0 < t < \infty, \\
V(0) = V_0
\end{cases}
\tag{3.3}
$$

in H.

Lemma 3.1. *The operator A is a positive definite self-adjoint operator on H with domain*

$$\mathcal{D}(A) = H_N^2([0, L]) := \{u \in H^2([0, L]) \text{ such that } \frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(L) = 0\}.$$

In addition, the domains of fractional powers $A^\theta, 0 \le \theta \le 1$, are characterized by

$$
\mathcal{D}(A^\theta) = \begin{cases}
H^{2\theta}([0, L]) & \text{for } 0 \le \theta < \frac{3}{4}, \\
H_N^{2\theta}([0, L]) & \text{for } \frac{3}{4} < \theta \le 1.
\end{cases}
$$

The lemma 3.1 is a special case of Theorem 16.7 and Theorem 16.9 in [22]. It then follows that A is a sectorial operator. The following theorem follows from Theorem 1.2, Theorem 1.4 and Theorem 1.5. It shows global existence and regularity of solutions to (3.3).

Theorem 3.2. *Assume that $V_0 \in \mathcal{D}(A^\beta)$ a.s. and $\mathbb{E}\|A^\beta V_0\| < \infty$.*

(i) *If $G \equiv 0$ on $[0, \infty)$ (i.e. (3.3) is a deterministic equation) and F satisfies (H3) for every $T > 0$, then the equation (3.3) has a unique continuous mild solution V on $[0, \infty)$ possessing the regularity:*

$$V \in \mathcal{C}((0, \infty); \mathcal{D}(A^{1-\alpha})),$$

$$A^\alpha V \in \mathcal{C}([0,\infty); H) \cap \mathcal{F}^{\beta-\sigma+\gamma,\gamma}((0,T]; H),$$

and satisfying the estimate

$$\|V(t)\| \leq \iota_\alpha B(\beta, 1-\alpha)\|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}}t^{\beta-\alpha} + \iota_0\|\xi\|$$

for every $T > 0$, $t \in [0,T]$ and $\gamma \in [0,\sigma]$.

(ii) *If $F \equiv 0$ on $[0,\infty)$ and G satisfies* (H4) *for every $T > 0$, then* (3.3) *has a unique mild solution V on $[0,\infty)$ possessing the regularity:*

$$V \in \mathcal{C}((0,\infty); \mathcal{D}(A^\nu)), \quad A^{\alpha_1}V \in \mathcal{C}^\gamma([\epsilon, T]; H) \qquad a.s.$$

and

$$\mathbb{E}\|A^{\alpha_1}V\| \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R}),$$

and satisfying the estimate

$$\mathbb{E}\|V(t)\| \leq C[\mathbb{E}\|A^\beta\xi\| + \|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}t^{\beta-\frac{1}{2}}], \qquad t \in [0,T]$$

for every $T > 0, \nu \in (0, \frac{1}{2}), \alpha_1 \in (0, \frac{1}{2} - \sigma], \gamma \in (0,\sigma)$ and $\epsilon \in (0,T]$, where C is some constant depending only on the exponents and constants $\iota_\theta, \upsilon_\theta$ ($\theta \geq 0$).

(iii) *If F and G satisfy* (H3) *and* (H4) *for every $T > 0$ and $\alpha \in (0, \frac{1}{2} - \sigma]$, then there exists a unique mild solution of* (1.12) *possessing the regularity:*

$$V \in \mathcal{C}((0,\infty); \mathcal{D}(A^\nu)), \quad A^\alpha V \in \mathcal{C}^\gamma([\epsilon, T]; H) \qquad a.s.$$

and

$$\mathbb{E}\|A^\alpha V\| \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R}),$$

and satisfying the estimate

$$\mathbb{E}\|V(t)\| \leq C[\mathbb{E}\|A^\beta\xi\| + \|A^{-\alpha}F\|_{\mathcal{F}^{\beta,\sigma}}\|G\|_{\mathcal{F}^{\beta,\sigma}((0,T];L_2(U;H))}t^{\beta-\frac{1}{2}}]$$

for every $t \in [0,T]$, $T > 0$, $\nu \in (0, \frac{1}{2}), \gamma \in (0,\sigma)$ and $\epsilon \in (0,T]$, where C is some constant depending only on the exponents and constants $\iota_\theta, \upsilon_\theta$ ($\theta \geq 0$).

Remark 4. A result in [18] shows that V is a Hölder continuous function with exponent $\frac{1}{4} - \epsilon$, for any $\epsilon > 0$. We have improved this result by showing that $A^\alpha V$ is Hölder continuous with an arbitrary exponent smaller than σ (noting that σ in Theorem 3.2 is possibly larger than $\frac{1}{4}$). Using the same framework, we can treat the equation (3.2) in higher dimensions and obtain similar results.

Acknowledgements

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